Classical singularities and semi-Poisson statistics in disordered systems

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We investigate a one-dimensional disordered Hamiltonian with a nonanalytical dispersion relation whose level statistics is exactly described by semi-Poisson statistics. It is shown that this result is robust, namely, it does not depend on the microscopic details of the Hamiltonian but only on the type of nonanalytical potential. We also argue that a deterministic kicked rotator with a steplike potential has the same spectral properties. Semi-Poisson statistics, typical of pseudointegrable billiards, have been frequently claimed to describe critical statistics, namely, the level statistics of a disordered system at the Anderson transition. However, we provide convincing evidence they are indeed different: each of them has its origin in a different type of classical singularity.

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I. INTRODUCTION

The properties of a quantum particle in a random potential are one of the most intensively studied problems in condensed matter physics since the landmark paper by Anderson [1]. According to the one-parameter scaling theory, in more than two dimensions, there exists a metal insulator transition, usually referred to as Anderson transition (AT), for a critical amount of disorder. Unfortunately, the AT in three and higher dimensions takes place in a region of strong disorder not directly accessible by current analytical techniques. Despite the lack of rigorous analytical results, it is by now well established, mainly through numerical simulations, that the AT is fully characterized by the level statistics and the anomalous scaling of the eigenfunction moments, $\mathcal{P}_q=\int d^d r|\psi(\mathbf{r})|^{2q}\propto L^{-D_q(q-1)}$ with respect to the sample size L, where D_q is a set of exponents describing the AT. Eigenfunctions with such a nontrivial (multi) scaling are usually dubbed multifractals [2]. Intuitively multifractality means that eigenstates have structures at many scales. Consider the volume of the subset of a box for which the absolute value of the wave function Ψ is larger than a fixed number $M\ll 1$. If this volume scales as L^{d^*} (with $d^*<d$), then d^* is called the fractal dimension $d^*<d$ of Ψ. In cases where the fractal dimension depends on the value of M, the wave function is said to be multifractal.

Critical statistics [3,4], the level statistics at the AT, are intermediate between Wigner-Dyson (WD) and Poisson statistics. Typical features include scale invariant spectrum [3], level repulsion, and asymptotically linear number variance [5].

Both level statistics and multifractal properties are universal in the sense that parameters such as the slope of the number variance or the set of multifractals exponents D_q depend only on the dimension of the system and not on boundary conditions, shape of the system, or the microscopic details of the disordered potential. However, the functional form of level correlators as the level spacing distribution or the number variance may be affected by such variables.

A natural question to ask is whether critical statistics can be reproduced by generalized random matrix models (RMM). The answer to this question is positive: Critical statistics have been found in RMM based on soft confining potentials [6], effective eigenvalue distributions [7,8] related to the Calogero-Sutherland model at finite temperature and random banded matrices with power-law decay [9]. The latter is especially interesting since the AT has been analytically established by mapping the problem onto a nonlinear σ model. All of the above RMM share the same spectral kernel in a certain range of parameters,

$$K(s)=T\frac{\sin(\pi s)}{\sinh(\pi sT)}$$

where T is a free parameter which enters in the definition of the above models.

In the context of quantum chaos, similarities with an AT have also been found in a variety of systems: Coulomb billiard [10], anisotropic Kepler problems [11], and generalized kicked rotors [12]. In all of them, the classical potential has a singularity and the classical dynamics is intermediate between integrable and chaotic. In a recent letter [13] we have put forward a new universality class in quantum chaos based precisely on the relation between the type of singularity of the classical potential and the properties of the quantum eigenstates. Specifically, for a certain kind of nonanalyticity, it was found that the level statistics is described by critical statistics and the eigenstates are multifractal as at the AT.

Similar properties have also been found in pseudointegrable billiards [14,15]. The level statistics of these models is very well described by a phenomenological short-range plasma model [14] whose joint distribution of eigenvalues is given by the classical Dyson gas with the logarithmic pairwise interaction restricted to a finite number k of nearest neighbors. Explicit expressions for the level statistics, usually referred to as semi-Poisson (SP) statistics, are available for general k. For $k=2$, $R_2(s)=1-e^{-4s}$, $P(s)=4se^{-2s}$, and $\Sigma^2(L)=L/2+(1-e^{-4L})/8$, where $R_2(s)$ is the two-level correlation function (TLCF), $P(s)$ is the level spacing distribution, and $\Sigma^2(L)$ is the number variance. Thus SP statistics reproduce typical characteristics of critical statistics as level repulsion, linear number variance, with a slope depending on k.

There are, however, quantitative differences; thus in critical statistics the joint distribution of eigenvalues can be considered as an ensemble of free particles at finite temperature with a nontrivial statistical interaction. The effect of a finite temperature is to suppress the correlations of distant eigenvalues. In SP this suppression is abrupt, in contrast to critical statistics, where the effect of the temperature is smooth.

The main aim of this paper is to describe under what conditions SP appears in disordered systems and quantum chaos and also to find out whether it belongs to the same universality class as critical statistics.

The organization of the paper is as follows. In the next section we introduce the models to be studied. In Sec. II we show analytically that in a certain range of parameters the level statistics of our disordered model is exactly given by SP. Numerical calculations of Sec. III further confirm this claim. Finally in Sec. IV we investigate the relation of SP statistics with a certain type of classical singularity.

II. THE MODELS

We start our investigation with a generalized kicked rotor with a steplike singularity,

$$\mathcal{H} = \frac{p^2}{2} + V(q)\sum_n \delta(t-nT), \tag{1}$$

with potential

$$V(q) = \begin{cases} \alpha, & \text{if } q \in [-\beta,\beta) \\ 0, & \text{otherwise}. \end{cases} \tag{2}$$

where $q\in[\pi,\pi)$, β sets the size of the step, and α is the height. We shall see in a broad range of parameters our results do not depend on the specific form of $V(q)$ but only on the presence of the steplike singularity.

The classical evolution over a period T is dictated by the map: $p_{n+1}=p_n-[\partial V(q_n)]/\partial q_n$, $q_{n+1}=q_n+Tp_{n+1}$ (mod 2π). The quantum dynamics is governed by the evolution operator $\mathcal{U}$ over a period T. Thus, after a period T, an initial state ψ_0 evolves to

$$\psi(T) = \mathcal{U}\psi_0 = e^{-i\hat{p}^2T/4\bar{h}}e^{-iV(\hat{q})/\bar{h}}e^{-i\hat{p}^2T/4\bar{h}}\psi_0, \tag{3}$$

where $\hat{p}$ and $\hat{q}$ stand for the usual momentum and position operator.

By using the method introduced in Ref. [16] one can map the evolution matrix Eq. (3) onto a one-dimensional (1D) Anderson model. We do not repeat here the details of the calculation but just state how the 1D Anderson model is modified by the nonanalytical potential. It turns out that the classical singularity induces long-range disorder in the associated 1D Anderson model,

$$\mathcal{H}\psi_n = \epsilon_n\psi_n + \sum_m F(n-m)\psi_m, \tag{4}$$

where $\epsilon_n \sim \tan(Tn^2)$ are pseudorandom numbers (provided $T \gg 1$) and [16]

$$F(m-n) = \int_{-\pi}^{-\pi} d\theta \tan(V(\theta))e^{-i\theta(m-n)} = A\frac{\sin\gamma(m-n)}{m-n},$$

with A, γ real positive constants easily obtained from β, α. The case $\gamma=\pi/2$ describes the high energy limit of the interaction of a charged particle with quenched Coulomb scatterers of alternate sign. The case of generic γ corresponds to a nontrivial charge distribution. We shall see that, indeed, the level statistics is to a great extent not sensitive to γ.

In this paper we mainly investigate the Hamiltonian Eq. (4) with the additional assumption that the ϵ_i's are truly random numbers extracted from a box distribution $[-W/2,W/2]$. We proceed so in order to make an accurate analysis of the level statistics necessary for a precise comparison with SP statistics. However, a detailed study of the Hamiltonian Eq. (1), including (multifractal) spectrum and quantum wave-packet evolution, will be published elsewhere [18].

III. ANALYTICAL RESULTS

We first state our main results:

(1) For $A \gg W$ and almost any γ in Eq. (4), the spectrum is scale invariant and the level statistics is exactly described by SP (after this work was completed, Bogomolny informed us that our results were in agreement with unpublished calculations by Schmit).

(2) There is a transition from SP to Poisson statistics as the strength of disorder is increased.

(3) In the region $A \gg W$ the eigenfunctions are multifractal but with a multifractal spectrum clearly different from the one observed at an AT.

We start by providing analytical evidence that the level statistics of Eq. (4) are described by SP statistics. We express the Hamiltonian (1) in Fourier space as

$$\mathcal{H} = E_k\psi_k + \sum_{k\neq k'} \hat{A}(k,k')\psi'_k,$$

where

$$E_k = \sum_r \frac{\sin\gamma r}{r}e^{ikr}$$

and

$$\hat{A}(k,k') = \frac{1}{N}\sum_n \epsilon_n e^{-in(k-k')}.$$

We fix $\gamma=\pi/2$ (our findings do not depend on γ). After a simple calculation we found that E_k is not a smooth function (this steplike singularity is indeed the seed for the appearance of SP), $E_k=A\pi/2$ for $k<\pi$, and $E_k=-A\pi/2$ for $k>\pi$. There are thus only two possible values of the energy separated by a gap $\delta=A\pi$. Upon adding a weak ($A\gg W$) disordered potential, this degeneracy is lifted and the spectrum is composed of two separate bands of size $\sim W$ around each of the bare points $-A\pi/2$ and $A\pi/2$. Since the Hamiltonian is invariant under the transformation $A\to -A$, the spectrum must also posses that symmetry. That means that, to leading order in A (neglecting $1/A$ corrections), the number of independent eigenvalues of Eq. (4) is $n/2$ instead of n.

We now show how this degeneracy affects the roots (eigenvalues) of the characteristic polynomial $P(t)=\det(H-tI)$. Let

$$P_{dis}(t)=a_0+a_1t+\cdots a_nt^n$$

be the characteristic polynomial associated with the disordered part of the Hamiltonian. We remark that, despite its complicated form, its roots, by definition, are random numbers with a box distribution $[-W,W]$. On the other hand, in the clean case,

$$P_{clean}(t)=(t-A)^{n/2}(t+A)^{n/2}$$

(π factors are not considered). Due to the $A\rightarrow -A$ symmetry, the full [Eq. (4)] case P_{full} corresponds with P_{dis} but replacing t^k factors by a combination $(t-A)^{k_1}(t+A)^{k_2}$ with $k_1+k_2=k$. The roots of P_{full} will be, in general, complicated functions of A. However, in the limit of interest, $A\gg W\rightarrow\infty$, an analytical evaluation is possible. By setting $t=t_1-A$ we look for roots t_1 of order the unity in the A band. We next perform an expansion of the characteristics polynomial P_{full} to leading order in A. Thus we keep terms $A^{n/2}$ and neglect lower powers in A. The resulting P_{full} is given by

$$P_{full}=t_1^{n/2}+\frac{a_{n-2}t_1^{n/2-2}}{3}+\frac{a_{n-3}t_1^{n/2-3}}{4}+\cdots 2\frac{a_{n/2+1}t_1}{n}+2\frac{a_{n/2}}{n+2}$$

where the coefficients a_n are the *same* as those of P_{dia} above but only $n/2$ of them appear in the full case. The eigenvalues ϵ_i' of Eq. (4) around the A band are $\epsilon_i'=A+\beta_i$ with β_i a root of P_{full}. The effect of the long-range interaction is just to remove all the terms with coefficients a_0 to $a_{n/2}$ from the characteristic polynomial of the diagonal disordered case. The spectrum is thus that of a pure diagonal disorder where half of the eigenvalues have been removed. The remaining eigenvalues are still symmetrically distributed [the ones with largest modulus are well approximated by $t_{max}=\pm\sqrt{(a_{n-2})/3}$] around A. That means, by symmetry considerations, that the removed ones must be either the odd or the even ones. This is precisely the definition of semi-Poisson statistics.

In conclusion, the power-law random banded matrices reproduce exactly the mechanism which is utilized in the very definition of SP. We finally mention that the only effect of the coefficients $3,4,\ldots,n+2/2$ is to renormalize the effective size of the spectrum: $\sim 2W$ for diagonal disorder and $\sim 2W/\sqrt{3}$ for Eq. (4).

IV. NUMERICAL RESULTS

The above analytical arguments have been fully corroborated by numerical calculations. By using standard diagonalization techniques we have obtained the eigenvalues of the Hamiltonian Eq. (4) for different volumes ranging from $N=500$ to $N=8400$. The number of different realizations of disorder was chosen such that for each N the total number of eigenvalues is at least 5×10^5, and in all cases $W=1$. Eigenvalues close to the band edges (around 20%) were discarded from the statistical analysis. The eigenvalues thus obtained were unfolded (by using the splines method) with respect to the mean spectral density. We first investigated the level statistics in the region $A\gg W$ where, according to the analytical findings above, SP holds. As shown in Fig. 1, the level spacing distribution $P(s)$ (including the tail in the inset) does not depend on the system size for volumes ranging from $N=500$ to $N=8400$. Moreover, level repulsion $P(s)\propto s$, $s\ll 1$ is still present, and the asymptotic decay of $P(s)$ (see inset) is exponential as for a insulator. All these features are spectral signatures of an AT.

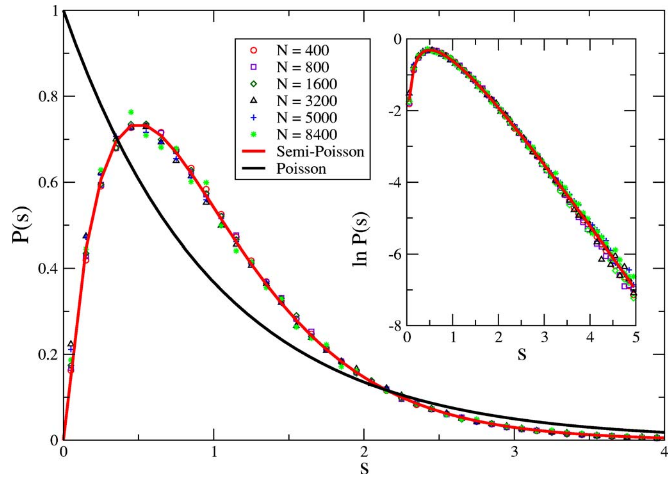

FIG. 1. (Color online) Level spacing distribution for $\gamma=\pi/2$ and $A=10$ and different system sizes. The agreement with SP is impressive even in the tail of $P(s)$ (see inset).

The study of long-range correlators as the number variance (see Fig. 2)

$$\Sigma^2(L)=\langle L^2\rangle-\langle L\rangle^2=\int_0^L\rho(s)ds-\int_0^L(L-s)R_2(s)ds,$$

where $\rho(s)$ is the spectral density, further confirms this point. It does not depend on the system size and its asymptotic

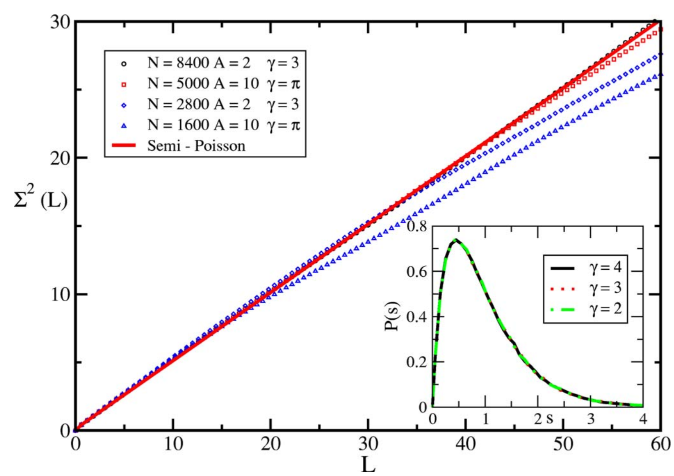

FIG. 2. (Color online) Number variance for different sizes N, A and γ. Provided that $A\gg1$ the number variance is given by SP statistics for any N, A, and γ. In the inset $P(s)$ is shown for $A=10$, $N=800$ and different γ. As shown $P(s)$ is not sensitive to the specific value of γ.

TABLE I. Set of multifractal dimensions D_q describing the AT for the Hamiltonian Eq. (4) with $A=10$, $\gamma=\pi/2$. The error in D_q is estimated to be around 10%.

q	1.5	2	2.5	3	4	5	6
D_q	0.36	0.3	0.28	0.26	0.24	0.22	0.22

behavior is linear $\Sigma^2(L) \sim L/2$ $L \gg 1$ as at the AT.

We now compare the level statistics of the Hamiltonian Eq. (4) with SP. As shown in Figs. 1 and 2 (inset) we could not detect any perceptible deviation from SP $P(s)=4se^{-2s}$ prediction for different γ's and $A \gg W$. The agreement is impressive even for the tail of $P(s)$ (inset). Also long-range correlators as the number variance follow the SP prediction

$$\Sigma^2(L) = \frac{L}{2} + \frac{1-e^{-4L}}{8}$$

for different parameter values (see Fig. 2). Deviations for small volumes are well-known finite size effects.

A remark is in order; although the above analysis clearly shows that the level statistics of our model are described by SP and share generic features of a disordered conductor at the AT, there are still important quantitative differences. Level statistics at the AT depend on the dimension of the space. For instance, in three-dimensions (3D) and four dimensions (4D) the slope of the number variance is 0.27 (0.41); by contrast SP predicts 1/2. Clear differences are also observed in short-range correlators as $P(s)$ [17,18] where it has been found that, despite their similarities, critical and SP statistics have different functional forms. The reason for such discrepancy follows.

As mentioned previously, level statistics at the 3D AT (critical statistics) are very accurately described by a generalized random matrix model [7] whose joint distribution of eigenvalues can be considered as an ensemble of free particles at finite temperature with a nontrivial statistical interaction. The statistical interaction resembles the Vandermonde determinant, and the effect of finite temperature is to suppress the correlations of distant eigenvalues. By contrast in SP, exclusively nearest neighbors eigenvalues are correlated.

We now investigate the eigenvector properties of the Hamiltonian Eq. (4). We shall see that, although they are multifractals, there exist important differences with respect to those at the AT.

We have studied the scaling of the eigenfunction moments P_q with respect to the sample size L. For multifractal wave functions,

$$\langle P_q \rangle = \int d^d r |\psi(\mathbf{r})|^{2q} \propto L^{-D_q(q-1)},$$

where the bracket stands for ensemble and spectral average and D_q is a set of exponents describing the transition. In Table I we show the multifractal dimensions D_q ($\pm 10\%$) for $A=10$, $\gamma=\pi/2$ obtained by numerical fitting of $\langle \log P_2 \rangle$: For small q, D_q depends clearly on q; however, for larger q the dependence is quite weak, suggesting that there may exist a

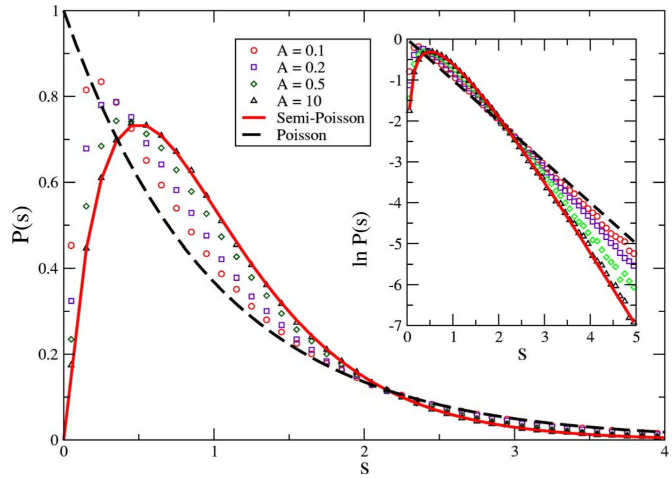

FIG. 3. (Color online) $P(s)$ for $\gamma=\pi/2$, $N=800$, and different A. A transition to Poisson statistics is observed in the limit $A \ll 1$.

critical q_c such that for $q > q_c$, $D_q \in [0,1]$ is a constant. Similar results have been recently reported for certain triangular billiards [15] though in this case it was claimed that D_q is constant for any q. We remark that, at the 3D AT, D_q depends explicitly on q for any q.

V. APPLICATION TO NONRANDOM HAMILTONIANS

We now investigate the original nonrandom Hamiltonian consisting of a kicked rotor with a nonanalytical steplike potential. From the above arguments it is clear that the Hamiltonian Eq. (1) avoids dynamical localization typical of a smooth potential due exclusively to the steplike singularity of the potential. In a recent paper [13], we found that certain types of classical singularities induce quantum power-law localization of the corresponding eigenvectors. For the case of log singularities, it was explicitly shown that eigenvectors were multifractals and the level statistics were given by critical statistics. It was also shown that these findings are universal in the sense that they do not depend on the microscopic details of the Hamiltonian but only on the classical singularity.

The results of this paper show that SP can be considered as the level statistics associated with chaotic systems with classical steplike singularities (in 1+1 dimensions) or with disordered systems with a steplike dispersion relation. We can thus unify different intermediate statistics (critical and SP statistics) in a broader classification based on the universal relation between classical singularities and level statistics features.

Finally we would like to discuss briefly two different issues. We have observed (see Fig. 3) that as A becomes comparable to W the level statistics shift slowly toward Poisson. The spectral correlations in this region are still scale invariant and even for $A \ll W$ small deviations from Poisson are not negligible. This finding suggests that the model is indeed critical for all values of γ and A. Another issue of interest is the robustness of our results under perturbations. We have added a flux to the Hamiltonian Eq. (4) in order to check whether the breaking of time-reversal invariance has any impact on the level statistics. The results are negatives; we have only observed the effect of the flux in the $s \to 0$ limit of the level spacing distribution $P(s) \sim s^2$ instead $P(s) \sim s$. This is consistent with previous results where it was shown that the effect of a magnetic flux gets diminished as the level statistics get closer to Poisson.

VI. CONCLUSIONS

We have introduced a class of systems with level statistics described by SP and multifractal eigenstates. The appearance of SP statistics has been related to steplike singularities of the classical potential and to a singular steplike dispersion relation in a disordered system. We have discussed similarities and differences with critical statistics and claimed that both are part of a larger classification scheme. Finally we have discussed the transition to Poisson in our model and the effect of a flux on the level statistics.

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